Inversion results and base-completeness for two approaches to proof-theoretic validity

Antonio Piccolomini d'Aragona
antonio.piccolomini@unisi.it

University of Siena
Aix-Marseille Université

Logic Seminar
Institute for Logic and Data Science
November 23rd 2023

General ideas of PTS

Tarskian semantics is inadequate for capturing the epistemic aspects of meaning and validity.

Our semantics should not be based on the realist notion of truth, but on the constructive notion of proof.

Intuitionism (BHK-semantics), verificationism (Dummett's theories of meaning), meaning as use (second Wittgenstein)...

Following some intuitions of Gentzen, proofs are understood as sound inferential structures in a Natural Deduction format.

Some rules are valid because they are meaning-constitutive. Others are valid when they can be justified in terms of the meaning-constitutive ones.

Aim of the talk

Original framework by [Prawitz 1973]: start from a notion of valid argument $\mathscr{D}$ on a deductive base-structure $\mathfrak{B}$ via suitably justified inferential structures. Then, $\Gamma \vDash^{\alpha}_{\mathfrak{B}} A$ iff there is $\mathscr{D}$ from Γ to A which is valid on $\mathfrak{B}$, while $\Gamma \vDash^{\alpha} A$ iff there is such a $\mathscr{D}$ which is valid over all $\mathfrak{B}$-s.

New mainstream approach [starting from Schroeder-Heister 2006]: just focus on consequence $\Gamma \vDash_{\mathfrak{B}} A$. All the constructive burden is put on $\mathfrak{B}$, and $\Gamma \vDash A$ means $\Gamma \vDash_{\mathfrak{B}} A$ for all $\mathfrak{B}$.

- Intuitively, $\vDash \subseteq \vDash^{\alpha}$ and $\vDash^{\alpha} \subseteq \vDash$, but this is not so straightforward.
- Many (in)completeness results for $\vDash$ [Piecha 2016 for an overview, while more recent results are Stafford 2021, Stafford & Nascimento 2023, Schroeder-Heister 2023]. Can they be adapted to $\vDash^{\alpha}$?

Gentzen's ND for ILP

$$\frac{\bot}{A}\,(\bot) \qquad \frac{A \qquad B}{A \wedge B}\,(\wedge_I) \qquad \frac{A_1 \wedge A_2}{A_i}\,(\wedge_{E,i}),\ i = 1,2$$

$$\frac{A_i}{A_1 \vee A_2}\,(\vee_{I,i}),\ i = 1,2 \qquad \frac{A \vee B \qquad \begin{matrix}[A]\\ \vdots \\ C\end{matrix} \qquad \begin{matrix}[B]\\ \vdots \\ C\end{matrix}}{C}\,(\vee_E)$$

$$\frac{\begin{matrix}[A]\\ \vdots \\ B\end{matrix}}{A \to B}\,(\to_I) \qquad \frac{A \to B \qquad A}{B}\,(\to_E) \qquad \neg A \overset{def}{=} A \to \bot$$

Gentzen's insight and Dummett's fundamental assumption

> *The introductions represent [...] the "definitions" of the symbols concerned, and the eliminations are no more, in the final analysis, than the consequences of these definitions. [Gentzen 1935]*

Dummett $\Rightarrow$ if A is provable, there is a proof of A ending by an introduction. Strengthened, every proof of A can be transformed into one ending by an introduction.

Gentzen may be referring to *derivations*, namely, formal objects in formal systems. Dummett decidedly refers to proofs, which can be assumed to live in no pre-determined system.

Prawitz's *inversion principle*

Inversion principle (Prawitz 1965)

Let α be an application of an elimination rule that has B as consequence. Deductions that satisfy the sufficient condition for deriving the major premise of α, when combined with deductions of the minor premises of α (if any), already "contain" a deduction of B; the deduction of B is thus obtainable directly from the given deductions without the addition of α.

Prawitz also refers to derivations. But the inversion principle will generalise to a semantic principle via Dummett's fundamental assumption + Prawitz's normalisation.

Local peaks in ND derivations

$$\dfrac{\dfrac{\begin{matrix}\mathscr{D}_1 \\ A_1\end{matrix} \qquad \begin{matrix}\mathscr{D}_2 \\ A_2\end{matrix}}{A_1 \wedge A_2}\,(\wedge_I)}{\begin{matrix}A_i \\ \mathscr{D}\end{matrix}}\,(\wedge_{E,i}) \qquad \dfrac{\dfrac{\begin{matrix}\mathscr{D}_i \\ A_i\end{matrix}}{A_1 \vee A_2}\,(\vee_{I,i}) \qquad \begin{matrix}[A_1] \\ \mathscr{D}_3 \\ B\end{matrix} \qquad \begin{matrix}[A_2] \\ \mathscr{D}_4 \\ B\end{matrix}}{\begin{matrix}B \\ \mathscr{D}\end{matrix}}\,(\vee_E)$$

$$\dfrac{\dfrac{\begin{matrix}[A] \\ \mathscr{D}_1 \\ B\end{matrix}}{A \rightarrow B}\,(\rightarrow_I) \qquad \begin{matrix}\mathscr{D}_2 \\ A\end{matrix}}{\begin{matrix}B \\ \mathscr{D}\end{matrix}}\,(\rightarrow_E) \qquad \dfrac{\dfrac{\begin{matrix}\mathscr{D}_1 \\ \bot\end{matrix}}{A \star B}\,(\bot) \qquad \dots}{\begin{matrix}C \\ \mathscr{D}\end{matrix}}\,(\star_E)$$

We require that $(\bot)$ is only applied with atomic conclusions.

Prawitz's reductions

$$\begin{matrix}\dfrac{\dfrac{\begin{matrix}\mathscr{D}_1\\ A_1\end{matrix}\qquad\begin{matrix}\mathscr{D}_2\\ A_2\end{matrix}}{A_1 \wedge A_2}\,(\wedge_I)}{A_i}\,(\wedge_{E,i})\\ \mathscr{D}\end{matrix} \quad\Longrightarrow\quad \begin{matrix}\mathscr{D}_i\\ A_i\\ \mathscr{D}\end{matrix}$$

$$\begin{matrix}\dfrac{\dfrac{\begin{matrix}\mathscr{D}_i\\ A_i\end{matrix}}{A_1 \vee A_2}\,(\vee_{I,i})\qquad\begin{matrix}[A_1]\\ \mathscr{D}_3\\ B\end{matrix}\qquad\begin{matrix}[A_2]\\ \mathscr{D}_4\\ B\end{matrix}}{B}\,(\vee_E)\\ \mathscr{D}\end{matrix} \quad\Longrightarrow\quad \begin{matrix}\mathscr{D}_i\\ [A_i]\\ \mathscr{D}_{i+2}\\ B\\ \mathscr{D}\end{matrix}$$

$$\begin{matrix}\dfrac{\dfrac{\begin{matrix}[A]\\ \mathscr{D}_1\\ B\end{matrix}}{A \to B}\,(\to_I)\qquad\begin{matrix}\mathscr{D}_2\\ A\end{matrix}}{B}\,(\to_E)\\ \mathscr{D}\end{matrix} \quad\Longrightarrow\quad \begin{matrix}\mathscr{D}_2\\ [A]\\ \mathscr{D}_1\\ B\\ \mathscr{D}\end{matrix}$$

Prawitz's normalisation theorem

Normal and non-normal derivations

$\mathscr{D}$ is *non-normal* iff it contains some local peaks. It is *normal* otherwise.

Normalisation theorem (Prawitz 1965)

If there is $\mathscr{D}$ for $\Gamma \vdash A$, then there is normal $\mathscr{D}^*$ for $\Gamma^* \vdash A$ with $\Gamma^* \subseteq \Gamma$.

Reducibility relation

$\mathscr{D} \leq \mathscr{D}^*$ iff $\mathscr{D}^*$ obtains from $\mathscr{D}$ by replacement of sub-arguments of $\mathscr{D}$ via Prawitz's reductions.

Reduction to normal form theorem (Prawitz 1971)

For every $\mathscr{D}$ for $\Gamma \vdash A$ there is (unique) normal $\mathscr{D}^*$ for $\Gamma^* \vdash A$ with $\Gamma^* \subseteq \Gamma$ such that $\mathscr{D} \leq \mathscr{D}^*$.

The normal-form theorem

Paths in derivations

A *path* in $\mathscr{D}$ is any branch of $\mathscr{D}$ which only passes through major premises of eliminations.

Normal form theorem (Prawitz 1965)

Every path in a normal $\mathscr{D}$ splits into three (possibly empty) parts:

- an E-part, where only eliminations are applied;
- a minimal part, where only $(\bot)$ is applied;
- an I-part, where only introductions are applied.

Fundamental corollary and invariance over atomic theories

Fundamental corollary

If $\mathscr{D}$ is normal for $\vdash A$, then $\mathscr{D}$ ends by an introduction.

ILP "confirms" Gentzen's claim, and "instantiates" Dummett's fundamental assumption. The idea is to generalise this towards a full-blooded semantics.

Systems over atomic theories (Prawitz 1971)

If Σ includes an atomic theory $\mathfrak{B}$, normalisation and fundamental corollary still hold, and the minimal parts in normal paths consist of $(\bot)$ plus rules from $\mathfrak{B}$.

Normalisation and its consequences require further proof-functions, e.g. *permutations* (for *maximal segments*, i.e., chains of $(\vee_E)$ starting from an introduction and ending into an elimination).

Back to semantics

Derivations = *formal* objects.

Proofs = valid arguments = *semantic* objects.

Normalisation shows that constructive systems "well-behave" under semantic insights. We may build a semantics by requiring that normalisation be, not *proved of a system*, but *assumed as a requirement*.

For doing this, we have to:

- generalise derivations;
- generalise reductions;
- introduce semantic structures for local evaluation.

Liberalising derivations: argument structures

Argument structures

An *argument structure* is a tree with formula-labelled nodes. The leaves are *assumptions*, the root the *conclusion*. Arches are *arbitrary inferences* (which may bind assumptions).

Open/closed, canonical/non-canonical

$\mathscr{D}$ is *closed* iff the set of the unbound assumptions $\Gamma = \varnothing$, it is *open* otherwise.
$\mathscr{D}$ is *canonical* iff it ends by introduction, it is *non-canonical* otherwise.

(Closed) instances

A (*closed*) *instance* of $\mathscr{D}$ is obtained by replacing every $B \in \Gamma$ with $\sigma(B)$, where σ associates B to a (closed) $\mathscr{D}^*$ for B.

Example of argument structure with instance

$$\dfrac{\dfrac{\overset{1}{[q\wedge r]} \qquad \neg(s\vee t)}{\neg\neg p} \qquad \dfrac{\dfrac{\overset{2}{[p\to(q\to t)]} \qquad p\vee\neg t \qquad \overset{3}{[t\vee s]}}{\neg p\vee\neg q}}{q\wedge s}\,3}{p\to q}\,1,2$$

Open non-canonical for $\neg(s\vee t), p\vee\neg t\vdash p\to q$.

$$\dfrac{\dfrac{\overset{1}{[q\wedge r]} \qquad \dfrac{\dfrac{\overset{4}{[p\to z]}}{s} \qquad \dfrac{}{p}}{\neg(s\vee t)}\,4}{\neg\neg p} \qquad \dfrac{\dfrac{\overset{2}{[p\to(q\to t)]} \qquad p\vee\neg t \qquad \overset{3}{[t\vee s]}}{\neg p\vee\neg q}}{q\wedge s}\,3}{p\to q}\,1,2$$

This is an instance [with $\sigma(p\vee\neg t) = p\vee\neg t$)].

Liberalising reductions: justifications

Inferences and inference rules

An *inference* is represented by

$$\frac{\mathscr{D}_1, ..., \mathscr{D}_n}{A}\,\delta$$

where δ is an assumptions-binding. An *inference rule* is a set of inferences.

Justifications of rules

A *justification* of R is a function ϕ defined on some $R^* \subseteq R$ such that, for every $\mathscr{D} \in R^*$:

- if $\mathscr{D}$ is from Γ for $A \Rightarrow \phi(\mathscr{D})$ is from $\Gamma^* \subseteq \Gamma$ for A;
- for every σ, $\phi(\mathscr{D}^\sigma) = \phi(\mathscr{D})^\sigma$.

Example of justifications

$$\dfrac{\dfrac{\begin{matrix}\mathscr{D}_1\\ A_i\end{matrix}}{A_1 \vee A_2} \qquad \begin{matrix}\mathscr{D}_2\\ \neg A_j\end{matrix}}{A_i}\,(\mathsf{DS}) \qquad \Longrightarrow \qquad \begin{matrix}\mathscr{D}_1\\ A_i\end{matrix}$$

$$\dfrac{\begin{matrix}\mathscr{D}_1\\ A \vee \neg B\end{matrix} \qquad \begin{matrix}1\\ [C, D]\\ \mathscr{D}_2\\ E \wedge F\end{matrix}}{B \to F}\,1 \qquad \Longrightarrow \qquad \dfrac{\begin{matrix}\mathscr{D}_1\\ A \vee \neg B\end{matrix} \qquad \dfrac{\dfrac{\overset{2}{[C]}}{E \vee \neg G} \qquad \begin{matrix}\mathscr{D}_3\\ \neg\neg A\end{matrix}}{D}\,2}{B \to F}$$

with $\mathscr{D}_3$ depending on at most the same assumptions as $\mathscr{D}_1, \mathscr{D}_2$.

Atomic bases

Level of a rule

The *level* of an atomic rule R, written $\mathfrak{L}(R)$, is:

- $R = A \in \texttt{ATOM} \Rightarrow \mathfrak{L}(R) = 0$;
- R is of the form

$$\dfrac{\begin{matrix}[\Gamma_1] & & [\Gamma_n] \\ A_1 & \ldots & A_n\end{matrix}}{B}$$

  where $\texttt{max}(\{\mathfrak{L}(\Gamma_i \triangleright A_i) \mid i \leq n\}) = \kappa \Rightarrow \mathfrak{L}(R) = \kappa + 1$.

Atomic base

An *atomic base* is a set of atomic rules. $\mathfrak{L}(\mathfrak{B}) = \texttt{max}\{\mathfrak{L}(R) \mid R \in \mathfrak{B}\}$. All bases contain explosion for atoms, i.e. $\bot \triangleright A$, for every $A \in \texttt{ATOM}$.

Examples of atomic bases

$$\frac{}{p} \qquad \frac{q \qquad p \qquad \begin{matrix}[r]\\ s\end{matrix}}{u} \qquad \frac{z}{q} \qquad \frac{r \qquad z}{s} \qquad \frac{u \qquad p}{r}$$

is an atomic base of level ≥ 2 (but strictly level 2). We have $z \vdash_{\mathfrak{B}} r$:

$$\dfrac{\dfrac{\dfrac{z}{q} \qquad \dfrac{}{p} \qquad \dfrac{\begin{matrix}1\\ [r]\end{matrix} \qquad z}{s}}{u}\,1 \qquad \dfrac{}{p}}{r}$$

Validity of an argument over a base

Validity of an argument over a base

$\langle \mathscr{D}, \mathfrak{J} \rangle$ is *valid* on $\mathfrak{B}$ *of level* n iff:

- $\mathscr{D}$ is closed $\Rightarrow$
  - the conclusion of $\mathscr{D}$ is atomic $\Rightarrow \mathscr{D} \leq_{\mathfrak{J}} \Delta \in \mathtt{DER}_{\mathfrak{B}}$;
  - the conclusion of $\mathscr{D}$ is logically complex $\Rightarrow \mathscr{D} \leq_{\mathfrak{J}} \mathscr{D}^*$ closed canonical whose immediate sub-arguments are valid on $\mathfrak{B}$;
- $\mathscr{D}$ is open $\Rightarrow \forall \sigma \ \forall \mu \in \Gamma \ \forall \mathfrak{C} \supseteq_n \mathfrak{B} \ \forall \mathfrak{J}^* \supseteq \mathfrak{J}$, if $\langle \sigma(\mu), \mathfrak{J}^* \rangle$ is valid on $\mathfrak{C}$, then $\langle \mathscr{D}^\sigma, \mathfrak{J}^* \rangle$ is valid on $\mathfrak{C}$.

Logical validity of arguments

$\langle \mathscr{D}, \mathfrak{J} \rangle$ is *valid of level* n iff, $\forall \mathfrak{B} \in \mathbb{B}^n$, $\langle \mathscr{D}, \mathfrak{J} \rangle$ is valid on $\mathfrak{B}$.

Some results 1

$\Gamma \vDash^{\alpha}_{\mathfrak{B},\ n} A$ = there is $\langle \mathscr{D}, \mathfrak{J} \rangle$ from Γ to A valid of level n on $\mathfrak{B}$.

$\Gamma \vDash^{\alpha}_{n} A$ = there is $\langle \mathscr{D}, \mathfrak{J} \rangle$ from Γ to A valid of level n.

Reducibility-consequence on base

(a) $A \in \mathtt{ATOM} \Leftrightarrow (\vDash^{\alpha}_{\mathfrak{B},\ n} A \Leftrightarrow \vdash_{\mathfrak{B}} A)$;

(b) $\vDash^{\alpha}_{\mathfrak{B},\ n} \bot \Leftrightarrow \forall A \in \mathtt{ATOM}\ (\vDash^{\alpha}_{\mathfrak{B},\ n} A)$;

(c) $\vDash^{\alpha}_{\mathfrak{B},\ n} A \wedge B \Leftrightarrow (\vDash^{\alpha}_{\mathfrak{B},\ n} A$ and $\vDash^{\alpha}_{\mathfrak{B},\ n} B)$;

(d) $\vDash^{\alpha}_{\mathfrak{B},\ n} A \vee B \Leftrightarrow (\vDash^{\alpha}_{\mathfrak{B},\ n} A$ or $\vDash^{\alpha}_{\mathfrak{B},\ n} B)$;

(e) $\vDash^{\alpha}_{\mathfrak{B},\ n} A \to B \Leftrightarrow A \vDash^{\alpha}_{\mathfrak{B},\ n} B$;

(f) $\Gamma \vDash^{\alpha}_{\mathfrak{B},\ n} A \Leftrightarrow \forall \mathfrak{C} \supseteq_n \mathfrak{B}\ (\Gamma \vDash^{\alpha}_{\mathfrak{C},\ n} A)$ [monotonicity]

(g) $\Gamma \vDash^{\alpha}_{\mathfrak{B},\ n} A \Rightarrow \forall \mathfrak{C} \supseteq_n \mathfrak{B}\ (\vDash^{\alpha}_{\mathfrak{C},\ n} \Gamma \Rightarrow \vDash^{\alpha}_{\mathfrak{C},\ n} A)$.

Some results 2

Validity equivalent to empty-base-validity

$\Gamma \vDash_n^\alpha A \Leftrightarrow \Gamma \vDash_{\mathfrak{B}^\varnothing,\ n}^\alpha A$.

Validity and base-validity

$\Gamma \vDash_n^\alpha A \Leftrightarrow \forall \mathfrak{B} \in \mathbb{B}^n\ (\Gamma \vDash_{\mathfrak{B},\ n}^\alpha A)$.

$\Gamma \vDash_n^\alpha A \Rightarrow \forall \mathfrak{B} \in \mathbb{B}^n\ (\vDash_{\mathfrak{B},\ n}^\alpha \Gamma \Rightarrow \vDash_{\mathfrak{B},\ n}^\alpha A)$.

Proof.

Via the above, monotonicity, point (g), and $\{\mathfrak{B} \mid \mathfrak{B} \supseteq_n \mathfrak{B}^\varnothing\} = \mathbb{B}^n$. □

Base-semantics

Base-consequence on a base

$\Gamma \vDash_{\mathfrak{B},\ n} A \Leftrightarrow$

- $A \in \texttt{ATOM} \Leftrightarrow \vdash_{\mathfrak{B}} A$;
- $\vDash_{\mathfrak{B},\ n} A \wedge B \Leftrightarrow \vDash_{\mathfrak{B},\ n} A$ and $\vDash_{\mathfrak{B},\ n} B$;
- $\vDash_{\mathfrak{B},\ n} A \vee B \Leftrightarrow \vDash_{\mathfrak{B},\ n} A$ or $\vDash_{\mathfrak{B},\ n} B$;
- $\vDash_{\mathfrak{B},\ n} A \to B \Leftrightarrow A \vDash_{\mathfrak{B},\ n} B$;
- $\Gamma \vDash_{\mathfrak{B},\ n} A \Leftrightarrow \forall \mathfrak{C} \supseteq_n \mathfrak{B}\ (\vDash_{\mathfrak{C},\ n} \Gamma \Rightarrow \vDash_{\mathfrak{C},\ n} A)$.

Base-validity

$\Gamma \vDash_n A \Leftrightarrow \forall \mathfrak{B} \in \mathbb{B}^n\ (\Gamma \vDash_{\mathfrak{B},\ n} A)$.

Some results

Monotonicity of base-consequence [Schroeder-Heister]

$\Gamma \vDash_{\mathfrak{B},\ n} A \Leftrightarrow \forall \mathfrak{C} \supseteq_n \mathfrak{B}\ (\Gamma \vDash_{\mathfrak{C},\ n} A)$.

Validity and base-validity [Schroeder-Heister]

$\Gamma \vDash_n A \Leftrightarrow \Gamma \vDash_{\mathfrak{B}^{\varnothing},\ n} A \Leftrightarrow \forall \mathfrak{B} \in \mathbb{B}^n\ (\vDash_{\mathfrak{B},\ n} \Gamma \Rightarrow \vDash_{\mathfrak{B},\ n} A)$.

(In)completeness

- For no n, IL is complete wrt $\vDash_n$ [de Campos Sanz, Piecha & Schroeder-Heister 2016, Piecha & Schroeder-Heister 2019]
- Inquisitive logic is complete wrt $\vDash_n$ with $n \geq 2$ [Stafford 2021]
- Completeness of IL obtains by "liberalising" the order relation on atomic bases [Stafford & Nascimento 2023, Schroeder-Heister 2023]

Sandqvist's variant

Sandqvist's base-semantics

$\Gamma \vDash^{s}_{\mathfrak{B},\ n} A \Leftrightarrow$

- $A \in \mathtt{ATOM} \Leftrightarrow\ \vdash_{\mathfrak{B}} A$;
- $\vDash^{s}_{\mathfrak{B},\ n} \bot \Leftrightarrow \forall A \in \mathtt{ATOM}\ (\vDash^{s}_{\mathfrak{B},\ n} A)$;
- $\vDash^{s}_{\mathfrak{B},\ n} A \wedge B \Leftrightarrow$ standard;
- $\vDash^{s}_{\mathfrak{B},\ n} A \vee B \Leftrightarrow$

  $\forall \mathfrak{C} \supseteq_{n} \mathfrak{B}\ \forall D \in \mathtt{ATOM}\ (A \vDash^{s}_{\mathfrak{C},\ n} D$ and $B \vDash^{s}_{\mathfrak{C},\ n} D \Rightarrow\ \vDash^{s}_{\mathfrak{C},\ n} D)$;
- $\vDash^{s}_{\mathfrak{B},\ n} A \rightarrow B \Leftrightarrow$ standard;
- $\Gamma \vDash^{s}_{\mathfrak{B},\ n} A \Leftrightarrow$ standard.

Sandqvist's validity

$\Gamma \vDash^{s}_{n} A \Leftrightarrow \forall \mathfrak{B} \in \mathbb{B}^{n}\ (\Gamma \vDash^{s}_{\mathfrak{B},\ n} A)$.

Some results

Monotonicity of Sandqvist's base-consequence

$\Gamma \vDash^{s}_{\mathfrak{B},\ n} A \Leftrightarrow \forall \mathfrak{C} \supseteq_n \mathfrak{B}\ (\Gamma \vDash^{s}_{\mathfrak{C},\ n} A)$.

Validity and base-validity

$\Gamma \vDash^{s}_{n} A \Leftrightarrow \Gamma \vDash^{s}_{\mathfrak{B}\varnothing,\ n} A \Leftrightarrow \forall \mathfrak{B} \in \mathbb{B}^n\ (\vDash^{s}_{\mathfrak{B},\ n} \Gamma \Rightarrow \vDash^{s}_{\mathfrak{B},\ n} A)$.

Sandqvist's completeness theorem

$\Gamma \vdash_{\mathrm{IL}} A \Leftrightarrow \Gamma \vDash^{s}_{2} A$.

In Sandqvist, $\bot$ is a nullary constant, and we do not have atomic explosion.

Inversion results 1

$$\forall \mathfrak{C} \supseteq_n \mathfrak{B}, \Gamma, A \ (\Gamma \vDash_{\mathfrak{C},\ n} A \Rightarrow \Gamma \vDash^{\alpha}_{\mathfrak{C},\ n} A). \quad (1)$$

$\vDash$ / $\vDash^{\alpha}$ Inversion

$\forall n, \mathfrak{B} \in \mathbb{B}^n \ ((1) \Rightarrow \forall \mathfrak{C} \supseteq_n \mathfrak{B}, \Gamma, A \ (\Gamma \vDash^{\alpha}_{\mathfrak{C},\ n} A \Rightarrow \Gamma \vDash_{\mathfrak{C},\ n} A))$.

Proof.

Induction if $\Gamma = \varnothing$, and relying on the closed case if $\Gamma \neq \varnothing$. (1) used in the implication case as (g) in Proposition on slide 20 *is not* a bi-implication. □

Corollary for $\mathfrak{B}^{\varnothing}$

$\forall n, \mathfrak{B} \in \mathbb{B}^n, \Gamma, A \ (\Gamma \vDash_{\mathfrak{B},\ n} A \Rightarrow \Gamma \vDash^{\alpha}_{\mathfrak{B},\ n} A) \Rightarrow \forall n, \mathfrak{B} \in \mathbb{B}^n, \Gamma, A \ (\Gamma \vDash^{\alpha}_{\mathfrak{B},\ n} A \Rightarrow \Gamma \vDash_{\mathfrak{B},\ n} A)$.

Inversion results 2

$$\forall \mathfrak{C} \supseteq_n \mathfrak{B}, \Gamma, A \ (\Gamma \vDash^s_{\mathfrak{C},\ n} A \Rightarrow \Gamma \vDash^\alpha_{\mathfrak{C},\ n} A). \quad (2)$$

$\vDash^s$ / $\vDash^\alpha$ Inversion

$\forall n, \mathfrak{B} \in \mathbb{B}^n \ ((1) \Rightarrow \forall \mathfrak{C} \supseteq_n \mathfrak{B}, \Gamma, A \ (\Gamma \vDash^\alpha_{\mathfrak{C},\ n} A \Rightarrow \Gamma \vDash^s_{\mathfrak{C},\ n} A)).$

Proof.

(2) is now also needed in the case of $\vee$ because of the elimination-like treatment of this constant. □

Corollary for $\mathfrak{B}^\varnothing$

$\forall n, \mathfrak{B} \in \mathbb{B}^n, \Gamma, A \ (\Gamma \vDash^s_{\mathfrak{B},\ n} A \Rightarrow \Gamma \vDash^\alpha_{\mathfrak{B},\ n} A) \Rightarrow \forall n, \mathfrak{B} \in \mathbb{B}^n, \Gamma, A \ (\Gamma \vDash^\alpha_{\mathfrak{B},\ n} A \Rightarrow \Gamma \vDash^s_{\mathfrak{B},\ n} A).$

Base-soundness and base-completeness

Let Σ be a recursive system and let $\Vdash$ be $\vDash, \vDash^s$ or $\vDash^\alpha$.

Base-soundness

Σ is *base-sound* over $\Vdash_n$ iff $\forall \mathfrak{B} \in \mathbb{B}^n, \Gamma, A \ (\Gamma \vdash_{\Sigma \cup \mathfrak{B}} A \Rightarrow \Gamma \Vdash_{\mathfrak{B},\ n} A)$.

Base-completeness

Σ is *base-complete* over $\Vdash_n$ iff $\forall \mathfrak{B} \in \mathbb{B}^n, \Gamma, A \ (\Gamma \Vdash_{\mathfrak{B},\ n} A \Rightarrow \Gamma \vdash_{\Sigma \cup \mathfrak{B}} A)$.

$\Sigma \cup \mathfrak{B}^{\varnothing} = \Sigma$.

From base to bases

Base-soundness implies soundness. Base-completeness implies completeness.

Base-soundness of ILP

For every n, ILP is base-sound over $\Vdash_n$.

Consequence of inversion on completeness

Let $\Vdash$ be $\vDash$ or $\vDash^s$.

Equivalence under existence of a base-sound-complete system

$\exists\Sigma$ (Σ base-complete on $\Vdash_n$ and base-sound on $\vDash_n^\alpha$) $\Rightarrow \forall\Gamma, A$ $(\Gamma \Vdash_n A \Leftrightarrow \Gamma \vDash_n^\alpha A)$.

Observe that this means: $\Vdash_n \subseteq \Sigma \subseteq \vDash_n^\alpha \Rightarrow \vDash_n^\alpha \subseteq \Vdash_n$.

Proof.

($\Leftarrow$) by base-soundness and base-completeness, $\forall n, \mathfrak{B} \in \mathbb{B}^n, \forall\Gamma, A$ $(\Gamma \Vdash_{\mathfrak{B},\ n} A \Rightarrow \Gamma \vDash_{\mathfrak{B},\ n}^\alpha A)$. Then apply inversion on $\Gamma \vDash_n^\alpha A$. □

Sufficient condition for completeness on $\vDash_n^\alpha$

$\forall\Sigma$(Σ base-complete on $\Vdash_n$ and base-sound on $\vDash_n^\alpha \Rightarrow \Sigma$ complete on $\vDash_n^\alpha$).

A strategy for reducibility-completeness?

ILP complete on $\vDash_2^s$ (and base-sound on $\vDash_n^\alpha$).

If, more strongly, ILP is also base-complete on $\vDash_2^s$, by the previous results, we have ILP also complete on $\vDash_2^\alpha$.

That would be very nice, since we could "extract" from a relation of logical consequence over all bases a logically valid argument witnessing that that relation holds.

But this strategy fails. Base-completeness of ILP is inconsistent at all levels. This is also nice.

Base as formulas

Base translation [Piecha, de Campos Sanz & Schroeder-Heister]

To any rule R we associate a set of disjunction-free formulas:

- $\mathfrak{L}(R) = 0 \Rightarrow R = A \in \mathtt{ATOM}$ and $R^* = R$
- $\mathfrak{L}(R) = k + 1 \Rightarrow R$ has the form

$$\dfrac{\begin{matrix}[\Gamma_1] & & [\Gamma_n] \\ A_1 & \ldots & A_n\end{matrix}}{A}$$

where $\mathfrak{L}(\Gamma_i \triangleright A_i) \leq k$ $(i \leq n)$, and $R^* = \bigwedge_{i \leq n}(\Gamma_i \triangleright A_i)^* \to A$.

$$\mathfrak{B} = \quad \dfrac{}{p} \qquad \dfrac{q \qquad r}{s} \qquad \dfrac{\begin{matrix}[t] & \\ u & \qquad w\end{matrix}}{z}$$

$\mathfrak{B}^* = \{p, (q \wedge r) \to s, ((t \to u) \wedge w) \to z\}$.

Export principle and GDP

Let $\Vdash$ be $\vDash$ or $\vDash^s$.

Export principle [Piecha, de Campos Sanz & Schroeder-Heister]

$\Vdash_n$ enjoys the *export principle* iff $\Gamma \Vdash_{\mathfrak{B},\ n} A \Leftrightarrow \Gamma, \mathfrak{B}^* \Vdash_n A$.

GDP [Piecha, de Campos Sanz & Schroeder-Heister]

$\Vdash_n$ enjoys the *generalised disjunction property* iff, for $\vee$ not occurring in Γ,

$$\Gamma \Vdash_{\mathfrak{B},\ n} A \vee B \Rightarrow (\Gamma \Vdash_{\mathfrak{B},\ n} A \text{ or } \Gamma \Vdash_{\mathfrak{B},\ n} B).$$

Incompleteness in base-semantics

GDP implies incompleteness [Piecha & Schroeder-Heister]

If GDP holds on $\Vdash_n$, then ILP is incomplete on $\Vdash_n$.

Proof.

GDP implies the logical validity of Harrop's rule. □

Export and completeness [Piecha & Schroeder-Heister]

Export implies incompleteness of ILP.

Proof.

Export plus completeness imply GDP. GDP implies incompleteness. □

Piecha and Schroeder-Heister work on $\vDash$. But their results can be extended to $\vDash^s$ since they only require validity of disjunction introduction, which holds for $\vDash^s$.

$$\mathfrak{B} = \dfrac{}{p} \qquad \dfrac{p}{v} \qquad \dfrac{q \qquad r}{z} \qquad \dfrac{\begin{matrix}[s]\\ u\end{matrix} \qquad v}{q}$$

$$\dfrac{\dfrac{\dfrac{[q\lor(t\to u)]^1 \quad \dfrac{[q]^2 \quad \dfrac{}{r}R_2}{z} \quad \dfrac{\dfrac{\dfrac{\dfrac{[s]^3}{t}R_1 \quad [t\to u]^4}{u} \quad \dfrac{\dfrac{}{p}}{v}}{q}3 \quad r}{z}}{z}2,4}{(q\lor(t\to u))\to z}1 \qquad \dfrac{}{r}R_2}{((q\lor(t\to u))\to z)\land r}$$

So: $r, R_1, R_2 \vdash_{\mathrm{ILP}\cup\mathfrak{B}} ((q\lor(t\to u))\to z)\land r$. Observe that

$$r, p, p\to v, q\land r\to z, s\to t, ((s\to u)\land v)\to q \vdash_{\mathrm{ILP}} ((q\lor(t\to u))\to z)\land r$$

where each assumption is ρ^* with $\rho \in \{R_1, R_2\}\cup\mathfrak{B}$.

Export principle in ILP 1

Extended ILP enjoys export

$\Gamma, \mathfrak{R} \vdash_{\mathtt{ILP} \cup \mathfrak{B}} A \Leftrightarrow \Gamma, \mathfrak{R}^*, \mathfrak{B}^* \vdash_{\mathtt{ILP}} A$.

$$\dfrac{\begin{matrix}[\Delta_1], \Gamma, \mathfrak{R} \\ \mathscr{D}_1 \\ B_1\end{matrix} \qquad \dots \qquad \begin{matrix}[\Delta_n], \Gamma, \mathfrak{R} \\ \mathscr{D}_n \\ B_n\end{matrix}}{A}\ R$$

Then $R^* = \bigwedge_{i \leq n}(\bigwedge_{\rho \in \Delta_i} \rho^* \to B_i) \to A$, and

$$\dfrac{\dfrac{\begin{matrix}\dfrac{[\bigwedge_{\rho \in \Delta_1} \rho^*]}{\Delta_1^*} \quad \Gamma, \mathfrak{R}^*, \mathfrak{B}^* \\ \mathscr{D}_1^* \\ B_1\end{matrix}}{\bigwedge_{\rho \in \Delta_1} \rho^* \to B_1} \qquad \dots \qquad \dfrac{\begin{matrix}\dfrac{[\bigwedge_{\rho \in \Delta_n} \rho^*]}{\Delta_n^*} \quad \Gamma, \mathfrak{R}^*, \mathfrak{B}^* \\ \mathscr{D}_n^* \\ B_n\end{matrix}}{\bigwedge_{\rho \in \Delta_n} \rho^* \to B_n}}{\bigwedge_{i \leq n}(\bigwedge_{\rho \in \Delta_i} \rho^* \to B_i)}$$

Export principle in ILP 2

$\lambda(\mathscr{D}) = 0, A \in \mathfrak{R}^* \cup \mathfrak{B}^* \Rightarrow$ by induction on the level of the rule R_A such that $R_A^* = A$. If $R_A \in \mathfrak{R}$ has level 0, this is trivial. Suppose R_A is

$$\dfrac{\begin{matrix}[\Delta_1] & & [\Delta_m] \\ B_1 & \dots & B_m\end{matrix}}{B}\, R_A$$

$A = R_A^* = \bigwedge_{i \leq m}(\bigwedge_{\rho \in \Delta_i} \rho^* \to B_i) \to B$ and, by i.h., for every $i \leq m$ and every $\rho \in \Delta_i$, $\rho \vdash_{\mathtt{IL} \cup \mathfrak{B}} \rho^*$. So

$$\dfrac{\begin{matrix}\Delta_i \\ \mathscr{D}_i \\ \bigwedge_{\rho \in \Delta_i} \rho^*\end{matrix} \qquad \dfrac{[\bigwedge_{i \leq m}(\bigwedge_{\rho \in \Delta_i} \rho^* \to B_i)]}{\bigwedge_{\rho \in \Delta_i} \rho^* \to B_i}}{B_i}$$

$(i \leq m)$. So, by applying R_A we get either $R_A \vdash_{\mathtt{ILP} \cup \mathfrak{B}} A$ or $\vdash_{\mathtt{ILP} \cup \mathfrak{B}} A$.

Export of ILP properly

$\Gamma \vdash_{\mathtt{ILP} \cup \mathfrak{B}} A \Leftrightarrow \Gamma, \mathfrak{B}^* \vdash_{\mathtt{ILP}} A$.

Base-completeness = export + completeness

Let $\Vdash$ be $\vDash$ or $\vDash^s$.

Base-completeness tantamount to export + completeness

ILP is base-complete on $\Vdash_n \Leftrightarrow \Vdash_n$ enjoys export and ILP is complete on $\Vdash_n$.

Proof.

$(\Rightarrow)$ $\Gamma \Vdash_{\mathfrak{B},\, n} A \Rightarrow \Gamma \vdash_{\mathtt{ILP} \cup \mathfrak{B}} A \Leftrightarrow \Gamma, \mathfrak{B}^* \vdash_{\mathtt{ILP}} A \Rightarrow \Gamma, \mathfrak{B}^* \Vdash_n A.$

$(\Leftarrow)$ $\Gamma \Vdash_{\mathfrak{B},\, n} A \Leftrightarrow \Gamma, \mathfrak{B}^* \Vdash_n A \Rightarrow \Gamma, \mathfrak{B}^* \vdash_{\mathtt{ILP}} A \Leftrightarrow \Gamma \vdash_{\mathtt{ILP} \cup \mathfrak{B}} A.$ □

General results on base-completeness

Inconsistency of base-completeness of ILP

For no n is ILP base-complete on $\Vdash_n$.

Proof.

Base-completeness $\Leftrightarrow$ export + completeness $\Rightarrow$ incompleteness. $\square$

Hence, although ILP is complete on $\vDash_2^s$, it is *not* base-complete on $\vDash_2^s$.

Conclusion: non-monotonic PTS

PTS can be given in a non-monotonic format, without requiring extensions of the atomic base in the open case.

Thus, when $\Vdash$ is $\vDash_{\mu}, \vDash_{\mu}^{s}$ or $\vDash_{\mu}^{\alpha}$, we have $\Gamma \Vdash_{\mathfrak{B},\ \mu,\ n} A$, but $\Gamma \nVdash_{\mathfrak{C},\ \mu,\ n} A$ for some $\mathfrak{C} \supseteq_{n} \mathfrak{B}$.

The inversion results and their consequences still hold.

Classical equivalence between $\vDash$ and $\vDash^{\alpha}$ on bases

$\forall n, \mathfrak{B} \in \mathbb{B}^{n}, \Gamma, A\ (\Gamma \vDash_{\mathfrak{B},\ \mu,\ n} A \Leftrightarrow \Gamma \vDash_{\mathfrak{B},\ \mu,\ n}^{\alpha} A)$, when the meta-language is classical.

References

- G. Gentzen (1935), *Untersuchungen über das logische Schließen*
- T. Piecha, W. de Campos Sanz & P. Schroeder-Heister (2016), *Failure of completeness in proof-theoretic semantics*
- T. Piecha & P. Schroeder-Heister (2019), *Incompleteness of intuitionistic logic with respect to proof-theoretic semantics*
- D. Prawitz (1965), *Natural deduction. A proof theoretical study*
- D. Prawitz (1971), *Ideas and results in proof theory*
- D. Prawitz (1973), *Towards a foundations of a general proof theory*
- T. Sandqvist (2015), *Base-extension semantics for intuitionistic sentential logic*
- P. Schroeder-Heister (2006), *Validity concepts in proof-theoretic semantics*
- W. Stafford (2021), *Proof-theoretic semantics and inquisitive logic*
- W. Stafford & V. Nascimento (2023), *Following all the rules: intuitionistic completeness for generalized proof-theoretic validity*

Thank you