Does every computably enumerable set admit a univocal Diophantine specification?

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Can we extract, from an available proof that each

$$D_a(x_1,\ldots,x_k) \quad = \quad 0\,, \qquad\qquad a\in\mathbb{N}\,,$$

in some indexed family of equations has at most one solution in $\mathbb{N}$, an effective bound $\mathscr{C}_a$ such that when $x_1 = \boldsymbol{v}_1 \quad ,\ldots,\quad x_k = \boldsymbol{v}_k$ solves $D_a = 0$ then $\boldsymbol{v}_1,\ldots,\boldsymbol{v}_k \leqslant \mathscr{C}_a$? In 1974, Yuri V. Matiyasevich provided a negative answer referring to a family of Diophantine equations that involve exponentiation, $u \mapsto 2^u$, and speculated that an alike limitation holds with D_a's forming a collection of polynomials over $\mathbb{Z}$.

Achieving the said limiting result amounts to proving that the graph

$$\mathcal{F}(a\,,\,b) \quad\Longleftrightarrow\quad F(a)=b$$

of any primitive recursive function $F:\mathbb{N}\longrightarrow\mathbb{N}$ can be specified in the form

$$\exists\,x_1\cdots\exists\,x_k \quad \varphi(\underbrace{a,b}_{\text{parameters}}\;,\;\underbrace{x_1,\ldots,x_k}_{\text{unknowns}}\,)\,,$$

so that *univocity*, to wit,

$$\exists\,x_1\cdots\exists\,x_k\,\forall\,y_1\cdots\forall\,y_k\;\left[\varphi(a,b,y_1,\ldots,y_k)\implies \&_{i=1}^{k}\big(y_i=x_i\big)\right]\,,$$

or (at worst) *finite-fold-ness*

$$\exists\,x\,\forall\,y_1\cdots\forall\,y_k\;\left[\varphi(a,b,y_1,\ldots,y_k)\implies \textstyle\sum_{i=1}^{k} y_i\leqslant x\right]\,,$$

holds, where φ is an arithmetic formula involving sum and product operators (along with positive integers and the logical connectives $=$, $\&$, $\vee$, $\exists\nu$) and devoid of universal quantifiers, negation, and implication.

As a preparatory measure towards this goal, one may consider surrogating the exponentiation operator by a relator $\mathscr{J}\subseteq\mathbb{N}^2$ subject to the constraints:

1. $\mathscr{J}(u\,,\,v) \implies v\leqslant u^u \;\&\; u>1$;
2. $\forall\,k\,\exists\,u\,\exists\,v\big[\,\mathscr{J}(u\,,\,v)\;\&\,u^k<v\,\big]$;
3. integers $\alpha>1$, $\beta\geqslant 0$, $\gamma\geqslant 0$, $\delta>0$ exist such that to each $w\in\mathbb{N}\setminus\{0\}$ there correspond u,v such that: $\mathscr{J}(u\,,\,v)$, $u<\gamma\,w^{\beta}$, and $v>\delta\,\alpha^{w}$ hold.

(These are the exponential-growth conditions proposed by Julia Robinson in 1952 with a strengthening suggested by Matiyasevich in 2010.)

To meet our desiderata, the relation designated by $\mathscr{J}$ should be representable in polynomial terms and should associate only a finite number of images v with each u in its domain. A promising pattern for constructing such a relation, advanced by Martin Davis in 1968, has been recently reused to construct five new candidate relations. Unfortunately, establishing whether at least one of the candidates is apt to the job calls for the hard task of proving that one of a few special quaternary quartic equations, which are

$$\begin{aligned}
2\cdot\big(r^2+2\,s^2\big)^2-\big(u^2+2\,v^2\big)^2 &= 1\,,\\
3\cdot\big(r^2+3\,s^2\big)^2-\big(u^2+3\,v^2\big)^2 &= 2\,,\\
7\cdot\big(r^2+7\,s^2\big)^2-3^2\cdot\big(u^2+7\,v^2\big)^2 &= -2\,,\\
11\cdot\big(r^2+r\,s+3\,s^2\big)^2-\big(v^2+v\,u+3\,u^2\big)^2 &= 2\,,\\
19\cdot 3^2\cdot\big(r^2+r\,s+5\,s^2\big)^2-13^2\cdot\big(v^2+v\,u+5\,u^2\big)^2 &= 2\,,\\
43\cdot\big(r^2+r\,s+11\,s^2\big)^2-\big(v^2+v\,u+11\,u^2\big)^2 &= 2
\end{aligned}$$

(each corresponding to a Heegner number), has a finite overall number of integral solutions.